Extended scaling for ferromagnetic Ising models with zero-temperature transitions

Helmut G. Katzgraber,[1] I. A. Campbell,[2] and A. K. Hartmann[3]
[1]*Theoretische Physik, ETH Zurich, CH-8093 Zurich, Switzerland*
[2]*Laboratoire des Colloïdes, Verres et Nanomatériaux, Université Montpellier II, 34095 Montpellier, France*
[3]*Institut für Physik, Carl-von-Ossietzky-Universtität, D-26111 Oldenburg, Germany*
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We study the second-moment correlation length and the reduced susceptibility of two ferromagnetic Ising models with zero-temperature ordering. By introducing a scaling variable motivated by high-temperature series expansions, we are able to scale data for the one-dimensional Ising ferromagnet rigorously over the entire temperature range. Analogous scaling expressions are then applied to the two-dimensional fully frustrated Villain model where excellent finite-size scaling over the entire temperature range is achieved. Thus we broaden the applicability of the extended scaling method to Ising systems having a zero-temperature critical point.

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I. INTRODUCTION

Studying the critical behavior of systems that order at zero temperature is challenging because the typically used Monte Carlo methods are generally unable to probe the critical behavior close enough to the zero-temperature critical point for traditional[1] finite-size scaling approaches to yield precise critical parameters. It is thus necessary to either incorporate scaling corrections[2] or find better approaches to scale the data.[3,4]

Using the intuition gained from high-temperature series expansions (HTSEs) a scaling approach has been introduced with the aim of extending the validity of critical scaling expressions to temperatures well above the critical region. So far, the approach has been applied to a number of model systems having finite ordering temperatures.[5–8] Inherently, this approach is ideal to study systems which order at zero temperature—such as a magnetic system *below* the lower critical dimension—since there only temperatures *above* the critical point can be accessed numerically. Thus, an important first step in adapting the extended scaling approach to these systems involves the appropriate choice of a scaling variable.

In this paper we derive extended scaling relations for two sample ferromagnetic models with no disorder in the interactions and which only order at zero temperature. First, we study the exactly solvable one-dimensional (1D) Ising ferromagnet and then use insights obtained to analyze the nontrivial two-dimensional (2D) fully frustrated Villain Ising model.[9] Data generated using Monte Carlo simulations for large system sizes and very low temperatures validate our scaling approach.

The paper is structured as follows. In Sec. II we discuss the extended scaling approach and how to adapt it to systems ordering at zero temperature, illustrating the results with the one-dimensional case in Sec. III. In Sec. IV we introduce the (Villain) fully frustrated Ising model and present details of our numerical calculations. The numerical data are then analyzed with the extended scaling approach followed by concluding remarks.

II. EXTENDED SCALING

Conventionally, at a continuous phase transition the power-law critical behavior of any observable $\mathcal{O}$ in the thermodynamic limit can be written as[1]

$$\mathcal{O}[t(T)] \sim t^{-y} \tag{1}$$

with $t=(T-T_c)/T_c$ as a scaling variable, T the temperature, and T_c the critical temperature at which the phase transition occurs. The exponent y describes the "strength" of the divergence at T_c. Alternatively, other critical variables which yield the same limiting behavior at criticality such as

$$\tau = (T - T_c)/T = 1 - \beta/\beta_c, \tag{2}$$

where $\beta=1/T$, can be used. For certain models further scaling variables have been introduced, e.g.,

$$\tau_s = \frac{1}{2}[\sinh^{-1}(2\beta) - \sinh(2\beta)] \tag{3}$$

for the 2D Ising ferromagnet[10–12] or

$$\tau_g = 1 - (\beta/\beta_c)^2 \tag{4}$$

for spin glasses with zero-mean symmetric interaction distributions.[5,13] All these scaling variables are proportional to $(T-T_c)$ for $T \to T_c$. However, T is not a sensible scaling variable for a zero-temperature ($T_c=0$) transition.[14] Cardy *et al.*[15] gave a renormalization-group technique rule for low-temperature limiting scaling variables at zero-temperature transitions. For example, for the Potts model studied by Cardy *et al.* the appropriate renormalization-group "temperature" scaling variable is proportional to $\exp(-\beta)$. By analogy, for a 1D ferromagnetic Ising model for which $T_c=0$ the scaling variable should be proportional to $\exp(-2\beta)$.[14]

Scaling expressions may also include temperature-dependent prefactors, which are noncritical but can be relevant in analyses that include a range of temperatures far above (or below) T_c. For instance, the reduced susceptibility $\chi(\beta)$ measured in numerical simulations (see below) is related to the thermodynamic susceptibility $\chi_{\mathrm{th}}(\beta)$ (which is the physically measurable observable) through $\chi(\beta) = \chi_{\rm th}(\beta)/\beta$. Thus, critical behavior of the form $\chi(\beta) \sim t^{-\gamma}$ implies $\chi_{\rm th}(\beta) \sim \beta t^{-\gamma}$, i.e., with a prefactor β.

The two thermodynamic limit observables that we discuss here are the ferromagnetic reduced susceptibility χ and the second-moment correlation length ξ. The ferromagnetic reduced susceptibility is given by

$$\chi = N\langle m^2\rangle, \tag{5}$$

where

$$m = \frac{1}{N}\sum_{i=1}^{N} S_i \tag{6}$$

is the magnetization per spin, N is the number of spins in the system and $\langle\cdots\rangle$ represents a thermal average. The second-moment correlation length is given by

$$\xi = \left[\frac{\mu_2}{z\chi}\right]^{1/2}, \tag{7}$$

where

$$\mu_2 = \sum_{i,j=1}^{N} r_{ij}^2\langle S_i S_j\rangle \tag{8}$$

is the second moment of the correlation function, z is the number of nearest neighbors, and r_{ij} is the distance between spins i and j.[16] For a hypercubic lattice $z=2D$, where D is the space dimension. Note that numerically we measure the finite-size correlation length [which is equivalent to the expression presented in Eq. (7)] as

$$\xi = \frac{1}{2\sin(|\mathbf{k}_{\rm m}|/2)}\left[\frac{\chi}{\chi(\mathbf{k}_{\rm m})} - 1\right]^{1/2}, \tag{9}$$

where $\mathbf{k}_{\rm m}=(2\pi/L,0)$ is the smallest nonzero wave vector (here in 2D) and $\chi(\mathbf{k})$ is the wave-vector-dependent reduced susceptibility

$$\chi(\mathbf{k}) = \frac{1}{N}\sum_{i,j=1}^{N}\langle S_i S_j\rangle e^{i\mathbf{k}\cdot\mathbf{r}_{ij}}. \tag{10}$$

HTSEs of the Ising ferromagnet in large space dimensions (i.e., in the mean-field regime)[7] show that simple relations for the reduced susceptibility, namely,

$$\chi(\beta) = \tau^{-1}, \tag{11}$$

and for the second-moment correlation length defined in Eq. (7)

$$\xi(\beta) = \beta^{1/2}\tau^{-1/2} \tag{12}$$

are exact for all $T > T_c = 1$. Thus, in this limit with the scaling variable τ [Eq. (2)], the critical power laws for the reduced observables $\chi_{\rm th}(\beta)/\beta$ and $\xi(\beta)/\beta^{1/2}$ hold exactly over the entire range of β from β_c to zero. In finite-dimensional ferromagnetic systems—if the same basic variables and expressions are used[5–7] with the modification necessary to give the right high-temperature limits—one obtains "extended scaling" equations in which the leading terms are

$$\chi(\beta) = C_\chi\tau^{-\gamma} + (1 - C_\chi) \tag{13}$$

and

$$\xi(\beta) = \beta^{1/2}[C_\xi\tau^{-\nu} + (1 - C_\xi)]. \tag{14}$$

In Eqs. (13) and (14) C_χ and C_ξ are critical amplitudes and γ and ν are the standard critical exponents.[1] With the appropriate critical parameters (critical temperature, critical exponents, as well as critical amplitudes) these expressions are exact *by construction* at the $\beta\to\beta_c$ and $\beta\to 0$ limits. Elsewhere, the expressions are not exact but have been shown to give good approximations to the true behavior for the entire paramagnetic temperature region. By introducing small correction terms these approximations can be improved considerably.

The expression for finite β_c given in Eqs. (13) and (14) cannot be used for systems with $T_c=0$ because $\beta_c=\infty$. In Sec. III we present an extended scaling approach tailored to systems having $T_c=0$ and a unique nearest-neighbor interaction strength $|J_{ij}|$ (in this case $|J_{ij}|=1$ $\forall$ i,j and no bond disorder).[17] We first present simple exact expressions for the 1D Ising ferromagnet for which the scaling variable

$$\tau_t(\beta) = 1 - \tanh(\beta) \tag{15}$$

works well. This is consistent with the Cardy *et al.* rule[15] because τ_t is equal to $2\exp(-2\beta)$ at low temperatures; but, like $(1-\beta/\beta_c)$, τ_t tends to 1 for $T\to\infty$. In the light of this result we then apply the same approach using τ_t to the nontrivial 2D fully frustrated Villain model. Our analysis shows that the extended scaling scenario with τ_t as a scaling variable gives an excellent account of the behavior of the correlation length and reduced susceptibility extrapolated to infinite size over the entire temperature range.

III. ONE-DIMENSIONAL ISING MODEL

To motivate the scaling expressions for ferromagnetic Ising models with zero transition temperature, we use as a toy model the one-dimensional Ising ferromagnet

$$\mathcal{H}_{\rm 1D} = -\sum_{i=1}^{L} J_{i,i+1}S_iS_{i+1} \tag{16}$$

with $J_{i,i+1}=1$ for all nearest neighbors i and $i+1$. The model orders only[14] at $T=0$ and expressions for $\xi(\beta)$ and $\chi(\beta)$ in the infinite-size limit are easily calculated from HTSE. The reduced susceptibility can be expanded as

$$\chi(\beta) = 1 + 2[\tanh(\beta) + \tanh^2(\beta) + \tanh^3(\beta) + \cdots] \tag{17}$$

and the second moment of the correlation is

$$\mu_2(\beta) = 2[\tanh(\beta) + 2^2\tanh^2(\beta) + 3^2\tanh^3(\beta) + \ldots]. \tag{18}$$

The second-moment correlation length is then given by Eq. (7) with $z=2$ in 1D. Using the mathematical identities

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}, \quad \sum_{n=1}^{\infty} n^2 x^n = \frac{(x+1)x}{(1-x)^3}, \tag{19}$$

the exact expressions for susceptibility and correlation length are thus

$$\chi(\beta) = \exp(2\beta) = \frac{2}{1-\tanh(\beta)} - 1 \tag{20}$$

and

$$\xi(\beta) = \frac{1}{2}[\exp(4\beta) - 1]^{1/2} = \frac{\tanh^{1/2}(\beta)}{1-\tanh(\beta)}. \tag{21}$$

Note that these expressions are valid for the *entire* temperature range.

Equations (20) and (21) are of the extended scaling form[7] [see Eqs. (13) and (14)] with τ_t [Eq. (15)] replacing τ in the extended scaling expressions for the ferromagnets with finite ordering temperatures. Finally, temperature-dependent effective exponents can be defined as

$$\gamma(\beta) = -d \log[\chi]/d \log[1-\tanh(\beta)] \tag{22}$$

and

$$\nu(\beta) = -d \log[\xi(\beta)/\tanh^{1/2}(\beta)]/d \log[1-\tanh(\beta)]. \tag{23}$$

In the limit $T \to T_c = 0$ the critical exponents are thus $\gamma_c = \nu_c = 1$.

For the Ising ferromagnet in one space dimension with linear extent $L=N$ the Fisher finite-size scaling rule[1] for an observable

$$\mathcal{O}(L,\beta) \sim L^{y/\nu}\mathcal{F}[L/\xi(\beta)] \tag{24}$$

when applied to the reduced susceptibility leads to

$$\frac{\chi(L,\beta)+1}{L/\tanh(\beta)^{1/2}} \sim \mathcal{F}_{\chi}\left[\frac{L[1-\tanh(\beta)]}{\tanh^{1/2}(\beta)}\right] \equiv \mathcal{F}'_{\chi}\left[\frac{\xi(\beta)}{L}\right]. \tag{25}$$

In Fig. 1 we illustrate the previously derived scaling relations with data for the reduced susceptibility for *finite* system sizes. The data are obtained by starting with the partition function $Z = \lambda_+^L + \lambda_-^L$ for a one-dimensional system of L spins in a field H,[18] with

$$\lambda_{\pm} = e^{\beta}[\cosh(\beta H) \pm \sqrt{\cosh^2(\beta H) - 2e^{-2\beta}\sinh(2\beta)}]. \tag{26}$$

To obtain the thermodynamic susceptibility $\chi_{\rm th}(\beta)$, we perform a second-order derivative of the free energy per spin $F = -(1/\beta)\ln Z(L)$ with respect to H, subsequently setting $H=0$. The raw data for $\chi = \chi_{\rm th}/\beta$ (inset) are scaled according to Eq. (25). The scaling is perfect.

IV. TWO-DIMENSIONAL VILLAIN MODEL

The two-dimensional fully frustrated Ising model, or Villain model,[9] consists of Ising spins on a square lattice with

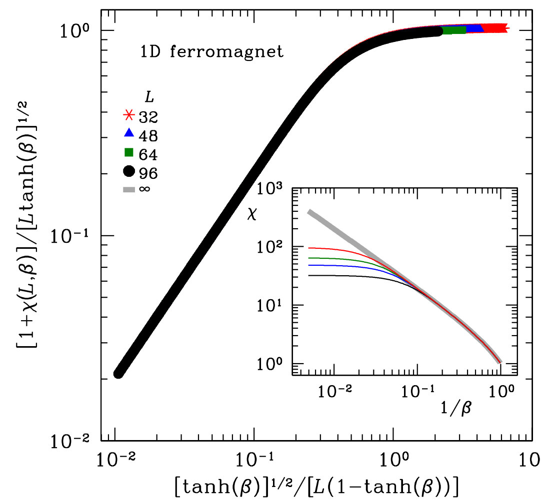

FIG. 1. (Color online) Scaled ferromagnetic reduced susceptibility [Eq. (5)] for the 1D Ising ferromagnet according to Eq. (25). The data scale perfectly and thus validate the derived scaling expressions. The inset shows the unscaled data for different system sizes, as well as the thermodynamic limit (thick gray line).

nearest-neighbor bonds $|J_{ij}|=1$; in the x direction all bonds are ferromagnetic, while in the y direction columns of bonds are alternately ferromagnetic and antiferromagnetic. The Hamiltonian is thus given by

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij}S_iS_j, \tag{27}$$

where $S_i = \pm 1$ represent Ising spins on a square lattice with $N=L^2$ spins. The system is fully frustrated; i.e., the product of the bonds around *each* plaquette of the system is negative,

$$\prod_{\square} J_{ij} = -1. \tag{28}$$

The model does not order at a finite temperature[19] but exhibits a critical point at zero temperature with diverging ferromagnetic reduced susceptibility and a ground-state degeneracy which grows exponentially with the system size.

For the scaling analysis we compute the reduced susceptibility [Eq. (5)] and the finite-size second-moment correlation length [Eq. (9)]. The simulations are done using exchange (parallel tempering) Monte Carlo[20–22] and the simulation parameters are presented in Table I. Equilibration is tested by a logarithmic binning of the data. Once the last three bins for all observables agree within error bars the system is considered to be in thermal equilibrium. We use periodic boundary conditions to reduce finite-size corrections.

Forgacs[19] showed analytically that the limiting low-temperature thermodynamic behavior of the correlation length of the 2D fully frustrated Villain model is strictly exponential, i.e., $\xi(\beta) \sim \exp(2\beta)$. Furthermore, the critical exponent η describing the decay of the correlation at T_c is exactly 1/2 such that in the low-temperature limit, using $\chi(\beta) \sim \xi(\beta)^{\gamma/\nu}$ [Eqs. (13) and (14)] and the standard scaling relation $\gamma = (2-\eta)\nu$, we obtain $\chi(\beta) \sim \xi(\beta)^{2-\eta} = \exp(3\beta)$. Based on an analysis of the size dependence of the energy by Lukic *et al.*[23] it has been conjectured that the low-

TABLE I. Parameters of the simulations. L denotes the system size, N_{sa} is the number of independent runs to improve the statistics and N_{sw} is the total number of Monte Carlo sweeps performed in a single run for each of the $2N_T$ replicas. T_{min} and T_{max} are the lowest and highest temperatures simulated, respectively, and N_T is the number of temperatures used in the parallel tempering method.

L	N_{sa}	N_{sw}	N_T	T_{min}	T_{max}
8	1000	131072	30	0.1	3.0
12	1000	131072	30	0.1	3.0
16	1000	131072	30	0.1	3.0
24	1000	262144	30	0.1	3.0
32	1000	262144	30	0.1	3.0
48	500	2097152	30	0.1	3.0
64	100	2097152	30	0.1	3.0
96	100	2097152	30	0.1	3.0

temperature limit for the correlation length is exactly $\xi(\beta)=(1/2)\exp(2\beta)$. No full HTSE study seems to have been carried out to date; however, by inspection, the leading HTSE terms for the reduced susceptibility are $\chi(\beta)=1+2\beta+O(\beta^2)$ and for the second moment of the correlation length $\mu_2=\beta+O(\beta^2)$.

Scaling dimensionless ratios of finite-size data for the correlation length $[\xi(2L,\beta)/\xi(L,\beta)]$ and susceptibility $[\chi(2L,\beta)/\chi(L,\beta)]$ vs the two-point finite-size correlation length divided by the system size $[\xi(L,\beta)/L]$, which is also a dimensionless quantity,[3] yields unique curves depending only on the universality class if there are no finite-size corrections to scaling. For the system sizes studied, the Villain model shows weak corrections to scaling. This can be seen in Figs. 2 and 3 where the ratios are shown as functions of $\exp[-L/\xi(L,\beta)]$.[3] In principle, it should be possible to use the ansatz of Calabrese *et al.*;[24] i.e.,

$$\frac{\xi(2L,\beta)}{\xi(L,\beta)} = \mathcal{F}[L/\xi(L,\beta)] + L^{-\omega}\mathcal{G}[L/\xi(L,\beta)], \qquad (29)$$

where ω is the finite-size scaling correction exponent and $\mathcal{F}$ and $\mathcal{G}$ are scaling functions. (Similar relations apply for the susceptibility χ.) As shown below when $T\to 0$, $\xi_\infty(\beta)$ diverges until $\xi_\infty(\beta)\gg L$; the values of the observables saturate at $\xi(L,\beta)\to\xi(L,\beta_c=\infty)$ and $\chi(L,\beta)\to\chi(L,\beta_c)$. The lowest temperature at which the simulations have been carried out is $T=0.1$. At this temperature we find $\xi(L=\infty,\beta)\sim 2.8\times 10^8$, thus for all L studied $\xi(L=\infty,\beta)\gg L$ and we can take the measured values of observables at all L as good approximations to the $T=0$ value. Hence, for $\beta\to\beta_c$ the prefactor $\sin(|\mathbf{k}_m|/2)$ is the only L-dependent factor in Eq. (9), which leads to $\xi(2L,\beta_c)/\xi(L,\beta_c)\to 2$. Furthermore, because $\eta=1/2$, $\chi(2L,\beta_c)/\chi(L,\beta_c)\to 2^{2-\eta}=2.82843\ldots$ exactly. Figure 4(a) shows $\chi(2L,\beta_c)/\chi(L,\beta_c)$ and Fig. 4(b) shows

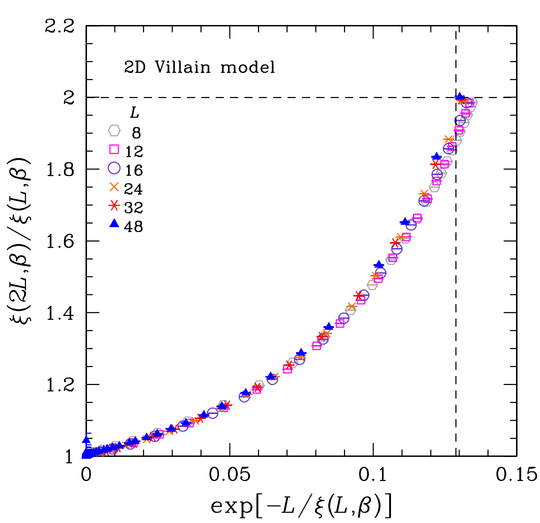

FIG. 2. (Color online) Scaling ratio $\xi(2L,\beta)/\xi(L,\beta)$ for the 2D Villain model as a function of $\exp[-L/\xi(L,\beta)]$ for different system sizes L. The dashed horizontal line corresponds to the exact infinite size value $\xi(2L,\beta_c)/\xi(L,\beta_c)=2$ at the critical point. The vertical line corresponds to the estimated infinite-size limit $\xi(\beta_c)/L=0.488$ (see text).

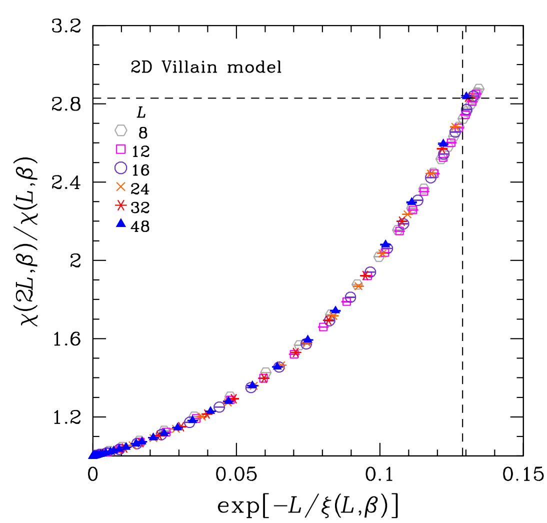

FIG. 3. (Color online) Scaling ratio $\chi(2L,\beta)/\chi(L,\beta)$ as a function of $\exp[-L/\xi(L,\beta)]$ for the 2D Villain model for different system sizes L. The dashed horizontal line corresponds to the exact infinite-size value $\chi(2L,\beta_c)/\chi(L,\beta_c)=2^{3/2}$ at the critical point since in general $\chi(T_c,L)\sim L^{2-\eta}$ and $\eta=1/2$. The vertical line corresponds to the estimated infinite-size limit $\xi(\beta_c)/L=0.488$ (see text).

$\xi(2L,\beta_c)/\xi(L,\beta_c)$ against $1/L$; it can be seen that the ratio behaves approximately as $\chi(2L,\beta_c)/\chi(L,\beta_c)\sim 2.8284 + 0.14/L$ [$\xi(2L,\beta_c)/\xi(L,\beta_c)\sim 2-0.24/L$] showing that the correction exponent for the leading correction at large L can be plausibly taken as $\omega\sim 1$ with further terms appearing at smaller L. In panel (c) of Fig. 4 we show data for $\chi(L,\beta_c)/L^{2-\eta}=\chi(L,\beta_c)/L^{3/2}$ against $1/L$. Fitting the data assuming $\omega=1$ gives the large-size limit $\chi(L,\beta_c)/L^{3/2} \approx 0.585(1)-0.05/L$. In a similar way we find the approximate limiting value of $\xi(L,\beta_c)/L\approx 0.488(1)+0.1/L$ [panel (d)] and of the Binder cumulant $g(L,\beta_c)\sim 0.691(1) + 0.10/L$ (not shown).

The analysis of the data at other temperatures is also consistent with a leading correction with an exponent $\omega\sim 1$ plus further correction terms for smaller L. At all temperatures studied the difference between the estimated infinite-size values for the observables and the measured large-L values are always less than 0.5% of the measured large-L values.

Inspired by the results on the 1D Ising ferromagnet outlined in Sec. III with $\tau_t=1-\tanh(\beta)$ as a scaling variable, we now test an analogous scaling of the data for the 2D fully frustrated Ising model. The critical exponents for the 2D fully frustrated Ising model [Eqs. (22) and (23)] are $\gamma_c = 3/2$, $\nu_c=1$, and $\eta_c=1/2$.[19] We thus construct trial expressions for the different observables as follows:

$$\chi_{\mathrm{FF}}(\beta) = C_\chi[1-\tanh(\beta)]^{-3/2} + (1-C_\chi) \qquad (30)$$

and

$$\xi_{\mathrm{FF}}(\beta) = \tanh^{1/2}(\beta)\left[\frac{C_\xi}{1-\tanh(\beta)} + (1-C_\xi)\right], \qquad (31)$$

where the critical amplitudes C_χ and C_ξ are the only adjustable parameters. It turns out that for the correlation length, the expression with $C_\xi=1.00(1)$, i.e.,

$$\xi_{\mathrm{FF}}(\beta)/\tanh^{1/2}(\beta) = [1-\tanh(\beta)]^{-1} \qquad (32)$$

gives an excellent overall fit to $\xi_\infty(\beta)/\tanh^{1/2}(\beta)$, which is the normalized infinite-size limiting curve estimated from the scaling curves; see Fig. 5. Over the entire temperature range the maximum difference between the fit and the numerical curve is approximately 0.5%. The expression in Eq. (32) with $C_\xi=1$ is identical to the exact expression for the 1D Ising ferromagnet. In the low-temperature limit with $C_\xi=1$, $\xi_{\mathrm{FF}}\to(1/2)\exp(2\beta)$ meaning that the present data and analysis are consistent with the Lukic *et al.* conjecture[23] within numerical precision. In the high-temperature limit $\xi_{\mathrm{FF}} \to \beta^{1/2}$, which is consistent with the first term of the HTSE for $\xi(\beta)$.

For the reduced susceptibility the fits to the numerical data for $\chi_\infty(\beta)$ with Eq. (30) indicate that C_χ is equal to 2.00(5); see Fig. 6. The fit in the higher-temperature range can be improved further by a correction term chosen, so that there is an exact agreement between the high-temperature limit obtained from the first two terms in the HTSE, namely, $\chi(\beta)=[1+2\beta^2+\cdots]$ as $\beta\to 0$. We thus obtain

$$\chi_{\mathrm{FF}}(\beta) = 2.0[1-\tanh(\beta)]^{-3/2} - 2.0 + [1-\tanh(\beta)]. \qquad (33)$$

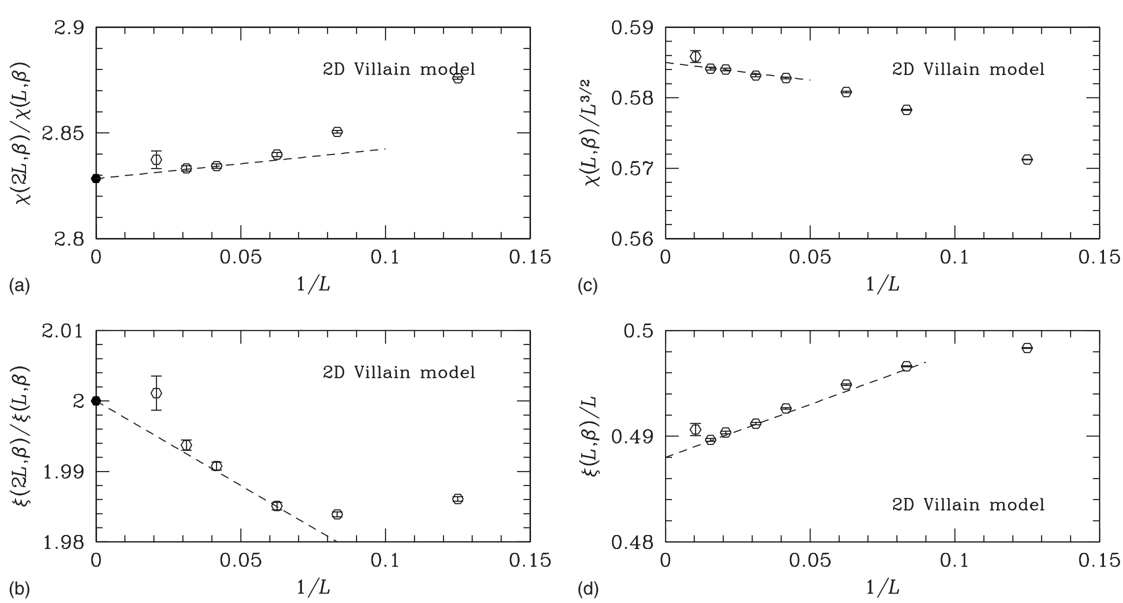

FIG. 4. (a) Scaling ratios $\chi(2L,\beta)/\chi(L,\beta)$ and (b) $\xi(2L,\beta)/\xi(L,\beta)$ plotted against $1/L$. The dashed line corresponds to $\chi(2L,\beta)/\chi(L,\beta)\sim 2^{3/2}+0.14/L$ [$\xi(2L,\beta)/\xi(L,\beta)\sim 2-0.24/L$]. The full symbol corresponds to the exact thermodynamic value $2^{3/2}$ in (a) and 2 in (b). (c) $\chi(L,\beta)/L^{3/2}$ vs $1/L$. The dashed line corresponds to $\chi(L,\beta)/L^{3/2}\sim 0.585-0.05/L$. Deviations appear for smaller values of L. (d) $\xi(L,\beta)/L$ plotted against $1/L$. The dashed line corresponds to $\xi(L,\beta)/L\sim 0.488+0.1/L$. The data thus suggest that a correction to scaling exponent $\omega\approx 1$ is plausible. All data are for $T=0.10$. Note that the data point for $L=96$ is generally a bit high possibly due to the small statistics used in the simulation.

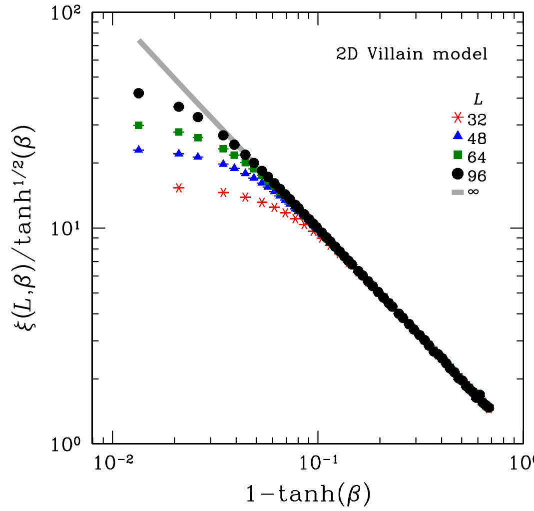

FIG. 5. (Color online) Normalized correlation length $\xi(\beta)/\tanh^{1/2}(\beta)$ for the 2D fully frustrated Ising model for different system sizes L as a function of $1-\tanh(\beta)$. The thick line corresponds to the extended scaling correlation length expression Eq. (32).

With the extended scaling expressions given above [Eqs. (32) and (33)] the standard Fisher finite-size scaling (FSS) [Eq. (24)] is modified (see Refs. 5 and 6 for details). For the finite-size correlation length we thus obtain from Eqs. (24) and (32)

$$\xi(L,\beta)/L \sim \mathcal{F}_{\xi}\left[\frac{L[1-\tanh(\beta)]}{\tanh^{1/2}(\beta)}\right] \equiv \mathcal{F}'_{\xi}\left[\frac{\xi_{\mathrm{FF}}}{L}\right], \quad (34)$$

whereas for the normalized reduced susceptibility we obtain from Eqs. (24), (32), and (33)

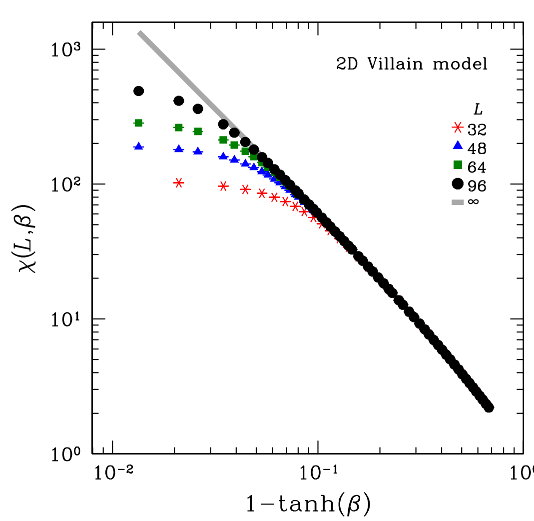

FIG. 6. (Color online) Data for the susceptibility $\chi(L,\beta)$ of the 2D fully frustrated Ising model for different system sizes L as a function of $1-\tanh(\beta)$. The thick line corresponds to the extended scaling expression in Eq. (33).

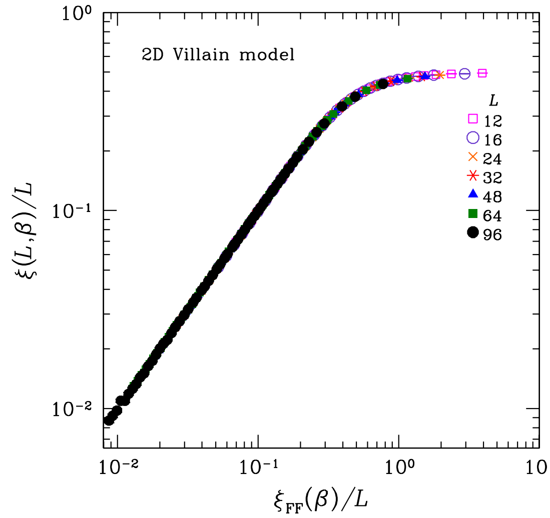

FIG. 7. (Color online) Finite-size scaling of the two-point correlation length data of the 2D fully frustrated Ising model using the extended scaling expression [Eqs. (32) and (34)].

$$\chi_n(L,\beta) \equiv \frac{\chi(L,\beta)+2-[1-\tanh(\beta)]}{[L/\tanh(\beta)^{1/2}]^{3/2}} \sim \mathcal{F}_{\chi}\left\{\frac{L[1-\tanh(\beta)]}{\tanh^{1/2}(\beta)}\right\} \equiv \mathcal{F}'_{\chi}\left[\frac{\xi_{\mathrm{FF}}}{L}\right]. \quad (35)$$

A finite-size scaling analysis of the data for the second-moment correlation length and the susceptibility using Eqs. (34) and (35) is shown in Figs. 7 and 8, respectively.

The scaling curves have a characteristic form. Quite generally, at small ξ_∞/L, $\xi(L,\beta)/L = \xi(L=\infty,\beta)/L$ so the log-log plot of, e.g., Fig. 7 is initially a straight line of slope 1 pass-

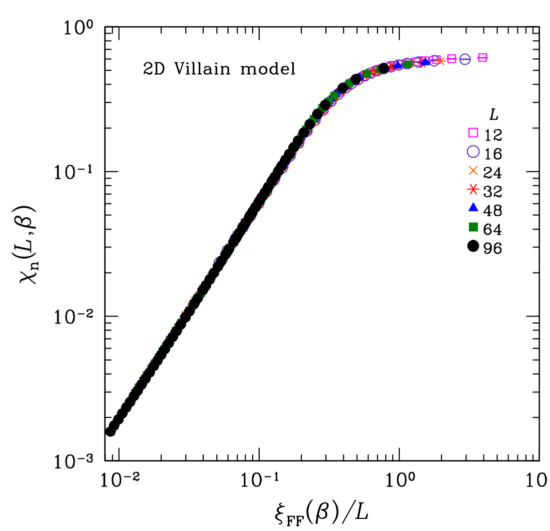

FIG. 8. (Color online) Finite-size scaling of the susceptibility of the 2D fully frustrated Ising model using the extended scaling expression for the normalized susceptibility $\chi_n(L,\beta)$, Eqs. (33) and (35).

ing through the point [1,1]. At the large ξ_∞/L limit (which is equivalent to $T=0$ for all L) the curves tend to plateau values $K_\chi=\chi(L)/L^{3/2}=0.585(1)$ and $K_\xi=\xi(L)/L=0.488(1)$ estimated above. If we ignore the marginal corrections to finite-size scaling, the crossover can be expressed phenomenologically as

$$\xi(L,\beta)/L = \left[\frac{[\xi_\infty(\beta)/L]^{z_\xi}}{1+(1/K_\xi)^{z_\xi}[\xi_\infty(\beta)/L]^{z_\xi}}\right]^{1/z_\xi}, \qquad (36)$$

where z_ξ is a crossover exponent. In the present case $z_\xi \approx 2.5$. For the reduced susceptibility the initial small-ξ_∞/L behavior is $\chi(L,\beta)/L^{2-\eta}\sim[\xi_\infty(\beta)/L]^{(2-\eta)}$ and the analogous phenomenological crossover equation is

$$\chi(L,\beta)/L^{2-\eta} = \Lambda\left[\frac{[\xi_\infty(\beta)/L]^{z_\chi(2-\eta)}}{1+(\Lambda/K_\chi)^{z_\chi}[\xi_\infty(\beta)/L]^{z_\chi(2-\eta)}}\right]^{1/z_\chi}, \qquad (37)$$

where z_χ is the crossover exponent, Λ is a constant, and K_χ is the plateau value. The phenomenological fit values in the present case are $z_\chi\approx 2.0$ and $\Lambda\approx 1.95$. For both observables the fits with crossover are of excellent quality.

V. SUMMARY AND CONCLUSION

We have presented scaling expressions motivated via high-temperature series expansions which extend the scaling functions across the whole temperature range for systems which order at zero temperature. For the 1D Ising ferromagnet we can derive exact extended scaling expressions of the form $\chi(\beta)=2\tau_t^{-1}-1$ and $\xi(\beta)=\tau_t^{-1}\tanh^{1/2}(\beta)$ with the scaling variable $\tau_t=1-\tanh(\beta)$, critical exponents $\gamma_c=\nu_c=1$ [defined via Eqs. (22) and (23)], and critical amplitudes $C_\chi=2$ and $C_\xi=1$.

From the insights gained from the study of the 1D ferromagnet, we use the same temperature variable τ_t to analyze numerical data for the 2D fully frustrated (Villain) model. The exact critical exponents are known:[19] $\gamma_c=3/2$ and $\nu_c=1$. We find that for the second-moment correlation length $\xi(\beta)=\tau_t^{-1}\tanh^{1/2}(\beta)$ with $C_\xi=1$ just as for the 1D ferromagnet. Furthermore, this result is consistent within numerical accuracy with the low-temperature-limit conjecture of Lukic *et al.*[23] that $\xi(\beta)\to(1/2)\exp(2\beta)$ for $T\to 0$; however, the present expression covers the entire temperature range. The approximate expression for the susceptibility of the Villain model [Eq. (33)] with critical amplitude $C_\chi=2.00(5)$ is in good agreement with the numerical data over the entire temperature range covered.

Summarizing, the temperature dependence of observables above a ferromagnetic transition (including $T_c=0$ transitions) can be written in terms of generic extended scaling forms[5–7] expressed to leading order as

$$\xi(x) = x^{-1/2}[C_\xi(1-x)^{-\nu}+(1-C_\xi)] \qquad (38)$$

and

$$\chi(x) = C_\chi(1-x)^{-\gamma}+(1-C_\chi) \qquad (39)$$

with the scaling variable x and critical parameters depending on the system studied. The expressions are exact by construction in the limits $\beta\to\beta_c$ and $\beta\to 0$ if the critical parameters are known. For a ferromagnet with $T_c>0$ $x=\beta/\beta_c$. Note that in the high-dimensional limit Eqs. (38) and (39) are exact.[7] In finite dimensions (but with nonzero T_c) the expressions remain as good approximations over the entire temperature range. For the two ferromagnets with $|J_{ij}|=1$, $T_c=0$, and no bond disorder, $x=\tanh(\beta)$ and thus $\tau_t=1-\tanh(\beta)$ replaces $(1-\beta/\beta_c)$ as the scaling variable. Effective exponents are defined through Eqs. (22) and (23). These relations are validated with numerical data on the 2D fully frustrated Ising model.

There are numerous possible candidate systems to which this approach should in principle be applicable *mutatis mutandis*. These include for instance the family of fully frustrated 2D systems studied by Forgacs,[19] the 2D three-state Potts antiferromagnet,[3,15] the 2D Ising antiferromagnet on a triangular lattice,[25] the 2D σ models,[3,26] as well as 2D Heisenberg models.[27] An interesting further step would be to determine scaling expressions for the two-dimensional bimodal Ising spin glass with $|J_{ij}|=1$ but with random signs for the interactions, which also orders at zero temperature. In that case the critical behavior of the model is highly controversial[28–33] and current data at finite temperature do not have the necessary quality within "traditional" scaling approaches to determine the true nature of the transition.

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